Wave Equation (d'Alembert solution)

the waves we've looked at are standing waves (not going anywhere)

but it's actually made of two traveling waves the move away from each other

this is also "visible" from the Fourier series solution

Problem A : $y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$

we know $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$\vdots$

$$y(x,t) = \underset{\substack{\uparrow \\ \text{each wave} \\ \text{is half magnitude}}}{\frac{1}{2}} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\underbrace{(x+at)}_{\substack{\text{wave} \\ \text{traveling LEFT}}}\right) + \underset{\uparrow}{\frac{1}{2}} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\underbrace{(x-at)}_{\substack{\text{wave} \\ \text{traveling RIGHT}}}\right)$$

d'Alembert solved the wave equation in 1747 (60 years before Fourier)

$y_{tt} = a^2 y_{xx}$ $\quad \underbrace{-\infty < x < \infty}_{\text{no BCs}}$, $t > 0$ $\quad$ a: speed of wave propagation

IC: $y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ " velocity

d'Alembert realized that an observer moving with one of the traveling waves would see a wave with constant magnitude

→ use as coordinate system

right-moving wave: wave speed is a

$$\frac{dx}{dt} = a \rightarrow x - at = \text{constant}$$

for left-moving wave: $x + at = \text{constant}$

let $\xi = x + at$

$\eta = x - at$

$y_{tt} = a^2 y_{xx}$ $\quad y(x,t)$

$y_x = y_\xi + y_\eta$ because $\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial y}{\partial \eta}\dfrac{\partial \eta}{\partial x}$

$y_t = a(y_\xi - y_\eta)$

$y_{xx} = y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta}$

$y_{tt} = a^2(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta})$

the wave becomes: $\boxed{y_{\xi\eta} = 0}$

integrate: $y_\eta(\xi,\eta) = \phi(\eta)$ (with respect ξ)

again: $y(\xi,\eta) = \phi(\eta) + \psi(\xi)$

back to x and t: $\boxed{y(x,t) = \underbrace{\phi(x-at)}_{\text{wave moving RIGHT}} + \underbrace{\psi(x+at)}_{\text{wave moving LEFT}}}$

initial conditions: $y(x,0) = f(x)$

$y_t(x,0) = g(x)$

$$y(x,t) = \phi(x-at) + \psi(x+at)$$

$$\boxed{f(x) = \phi(x) + \psi(x)}$$

$$y_t(x,t) = \frac{\partial \phi}{\partial(x-at)} \frac{\partial(x-at)}{\partial t} + \frac{\partial \psi}{\partial(x+at)} \frac{\partial(x+at)}{\partial t}$$

at $t=0$

$$g(x) = \phi'(x) \cdot -a + \psi'(x) \cdot a$$

$$\boxed{g(x) = -a\phi'(x) + a\psi'(x)}$$

integrate this from x_0 to x

$$\boxed{-a\phi(x) + a\psi(x) = \int_{x_0}^{x} g(s)\,ds}$$

Solve 1st and 3rd eqs simultaneously

⋮

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds$$

$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds$$

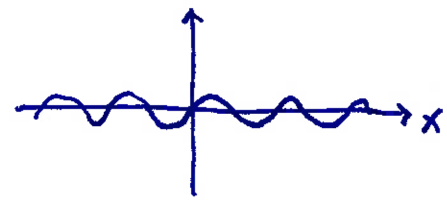
Sub into $y(x,t) = \phi(x-at) + \psi(x+at)$

$\vdots$

$$\boxed{y(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds}$$

<u>example</u> $f(x) = \sin(x)$ initial displacement

$g(x) = 0$ initial velocity



$a = 1$

$$y(x,t) = \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right]$$

half of initial displacement to the right

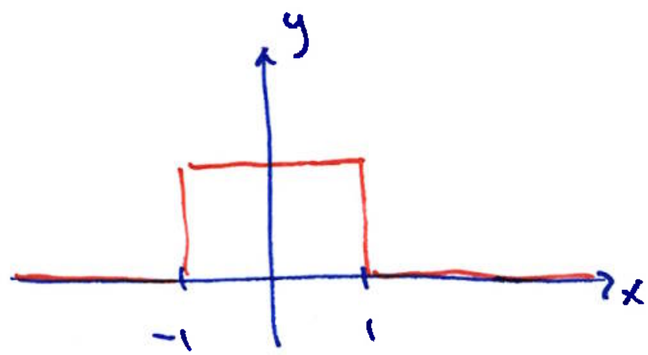
half of initial displacement to the left example

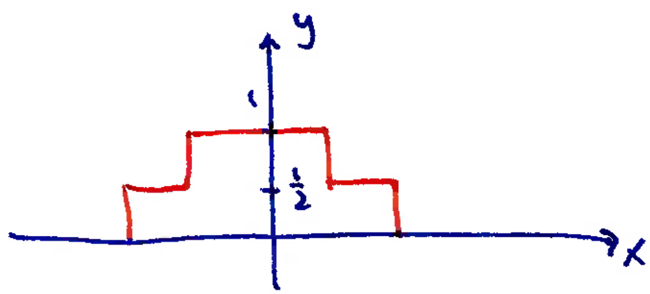
$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{else} \end{cases}$$

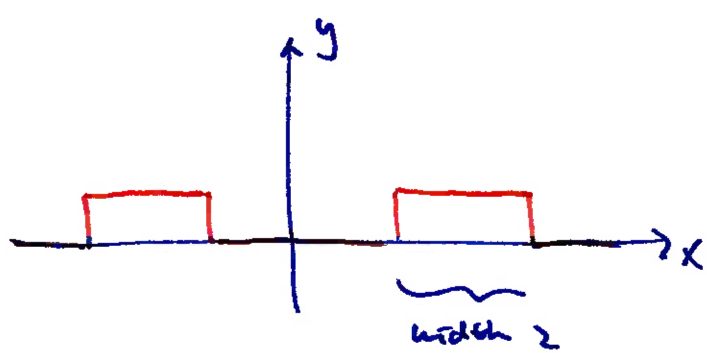
$g(x) = 0$

$a = 1$

$$y(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right]$$



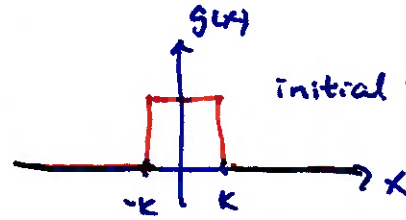




example

$f(x) = 0$

$$g(x) = \begin{cases} V_0 & \text{if } |x| < K \\ 0 & \text{else} \end{cases}$$

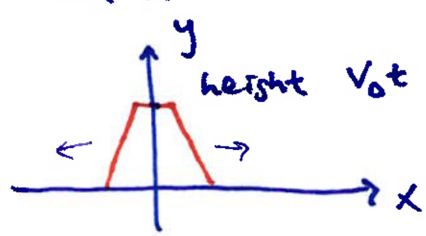


initial velocity (hammered string)

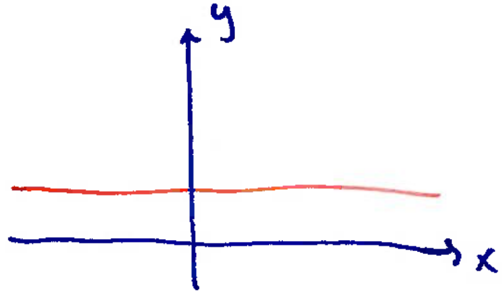
$a = 1$

$$y(x,t) = \underbrace{\frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds}_{\text{area under } g(x) \text{ on } x-t < x < x+t}$$

if $t < K$



if t is large



does not return to 0 !